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LOCAL PRESSURE OSCILLATION AND BOUNDARY TREATMENT FOR THE 8-NODE QUADRILATERAL

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SUMMARY

The 8-node (serendipity) velocity basis with C^0 bilinear pressure is a popular element but has been observed to yield poor pressures. We present some details of numerical experiments that indicate the local nature of the error and the effects of mesh refinement, increasing Reynolds number and regularity of the data. This leads to a strategy for appropriately modifying the data near the corners that is effective in improving the computed pressure approximation.

KEY WORDS Pressure Oscillations Driven Cavity Boundary Data

DISCUSSION

The eight-node (serendipity) quadrilateral element is used extensively in finite element computations since it offers the usual advantages of the 9-node (biquadratic) element, yet does not employ an internal node and associated degrees of freedom. We are specifically interested here in the use of this element for C^0 approximation of the velocity field, together with C^0 bilinear approximation of the pressures in a mixed finite element method for viscous flow computation. The observations apply equally, however, to the incompressible elasticity problem and possibly to other coupled systems of equations of similar structure.

The pressures computed using the eight-node element have been observed in practice to be significantly poorer than those obtained with the 9-node element.[1,2] Our purpose here is to follow up this prior work and present some detailed numerical results indicating the behaviour of the solutions obtained with this element for the familiar driven cavity problem. The driven cavity is particularly interesting in the context of the present study since the corner singularities strongly influence the pressure field and amplify the effect of the element. We examine the behaviour of the pressure approximation with mesh refinement and also consider the approximation at higher Reynolds numbers. These results lead us to examine the effect of smoothing the data in a specific manner—namely, to satisfy the mass conservation condition at the corners.

NUMERICAL STUDIES AND ANALYSIS

We consider the 'non-leaky' driven unit cavity and uniform meshes of 5×5, 10×10 and 20×20 elements. Two variants of the standard approximation to the boundary conditions are first considered: (i) the lid velocity is taken to vary linearly from zero at the corners $x = 0$, $x = 1$ to unity on the interval $0{\cdot}2 \leqslant x \leqslant 0{\cdot}8$ for all meshes, so that the same boundary-value problem is being

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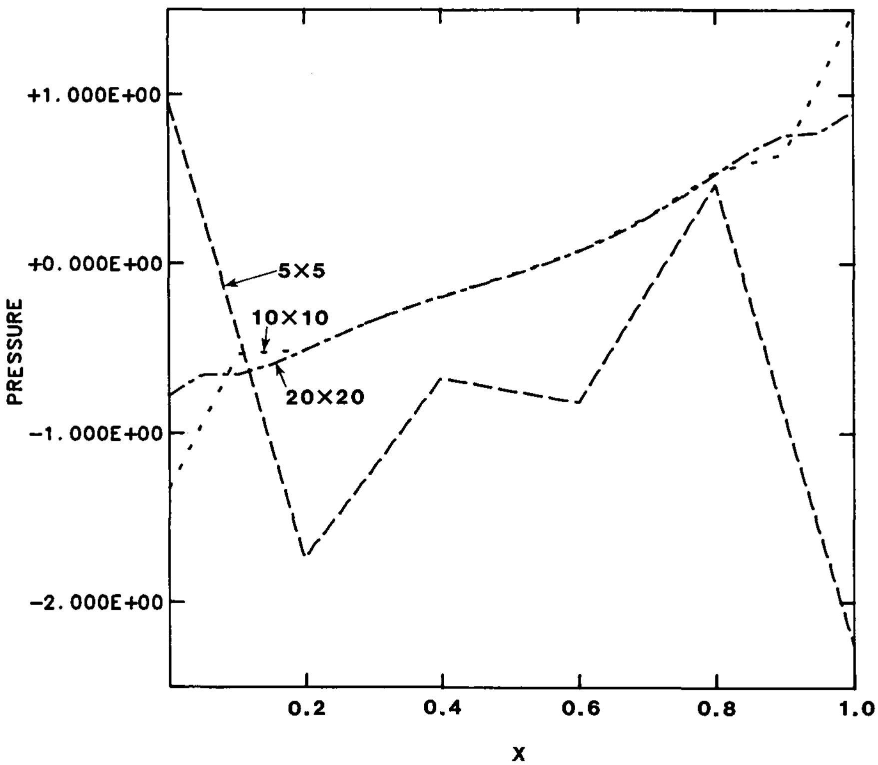


Figure 1. Pressure profiles at $y = 0{\cdot}8, 0 \leqslant x \leqslant 1$ in the unit cavity for 8-node element and data type (i) on $y = 1$ (i.e. $u = 1$ for $0{\cdot}2 \leqslant x \leqslant 0{\cdot}8$, otherwise u linear). Results are for uniform meshes of 5×5, 10×10 and 20×20 elements

solved in each case; (ii) the lid velocity changes linearly over the subintervals corresponding to the element sides adjacent to the corners $x = 0$ and $x = 1$. In this later case, the approximation to the 'true' discontinuity is closer as the mesh is refined.

The pressure profiles* at $Re = 10$ along the section $y = 0{\cdot}8$, $0 \leqslant x \leqslant 1$ are plotted in Figure 1 for solutions on uniform (5×5), (10×10) and (20×20) grids using the 8-node element and the first variant of the boundary condition ('fixed slope'). Pressure values on the coarsest mesh are very poor and oscillatory. Results using the 10×10 and 20×20 meshes are in good agreement throughout the interior but differ locally near the boundary where the oscillation is confined. Similar results are obtained with the second form of the boundary condition ('varying slope'), the amplitude of the pressure error near the boundary being larger as one might anticipate (Figure 2). The profiles for the 8-node and 9-node elements are compared in Figure 3 for the (5×5) and (10×10) grids. The 9-node results do not exhibit this local oscillatory behaviour and are quite acceptable even on the coarse (5×5) grid. The horizontal velocity component along this section is plotted in Figure 4. There is a slight oscillation in the velocity for the very coarse mesh, but these results are uniformly close.

The effect of increasing the strength of the non-linearity was next considered. In Figure 5, we plot pressures at the same section for solutions obtained at $Re = 800$ using the 8-node element.

* These results at very low Reynolds number were observed to be graphically indistinguishable from those for Stokes flow ($Re = 0{\cdot}0$)

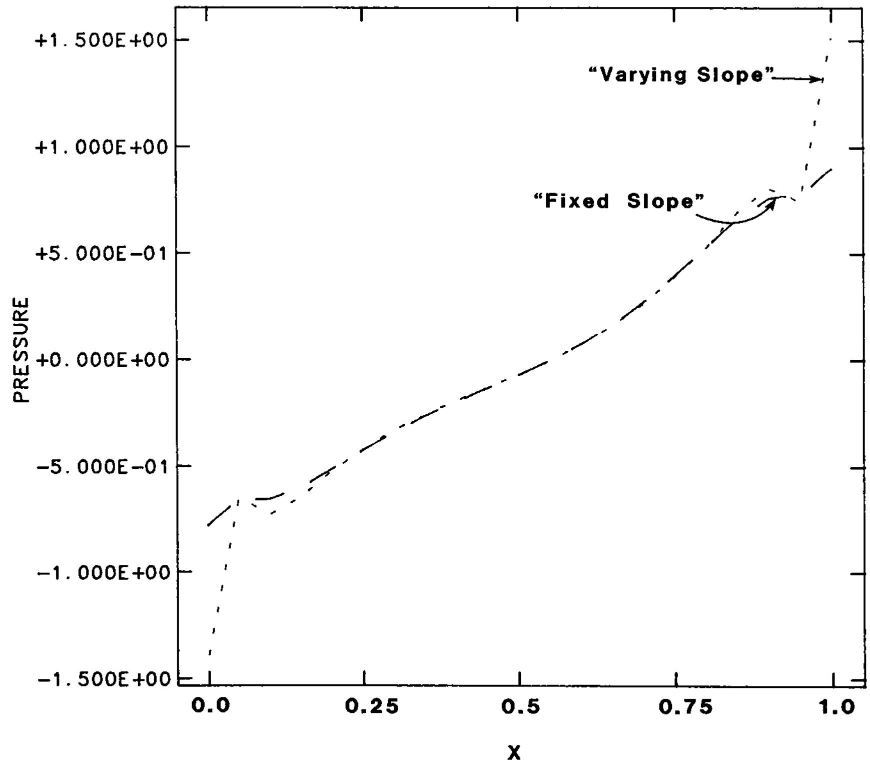


Figure 2. Comparison of pressure profiles at $y = 0{\cdot}8, 0 \leqslant x \leqslant 1$ for 8-node element and 20×20 mesh using type (i) data ($u = 1$ on $0{\cdot}2 \leqslant x \leqslant 0{\cdot}8$, $y = 1$) and type (ii) data ($u = 1$ on $0{\cdot}05 \leqslant x \leqslant 0{\cdot}95$)

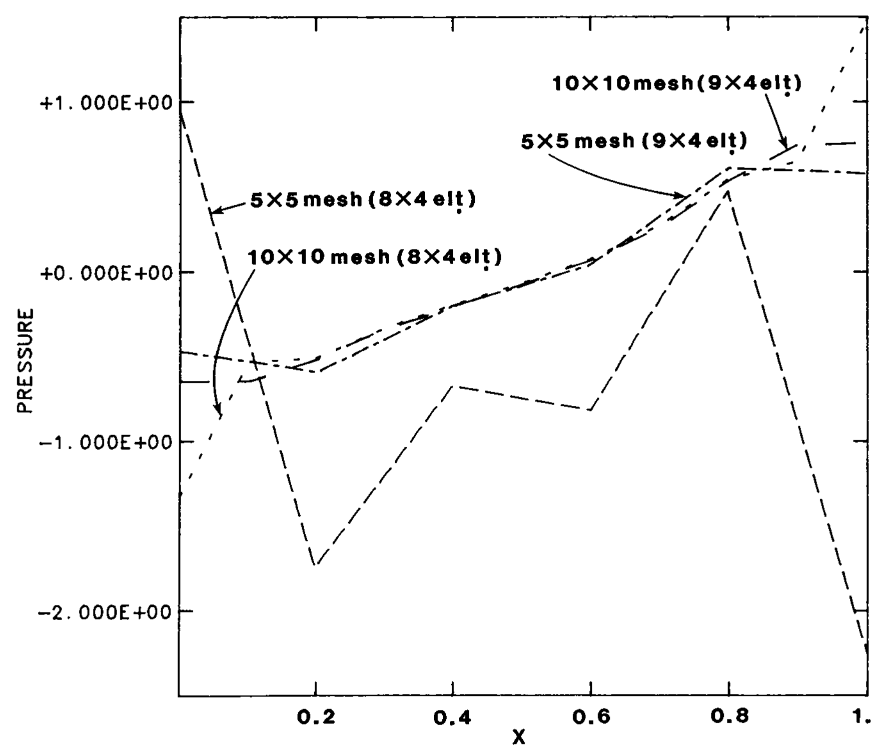


Figure 3. Comparison of section pressures for 5×5 and 10×10 meshes using the 8-node and 9-node elements

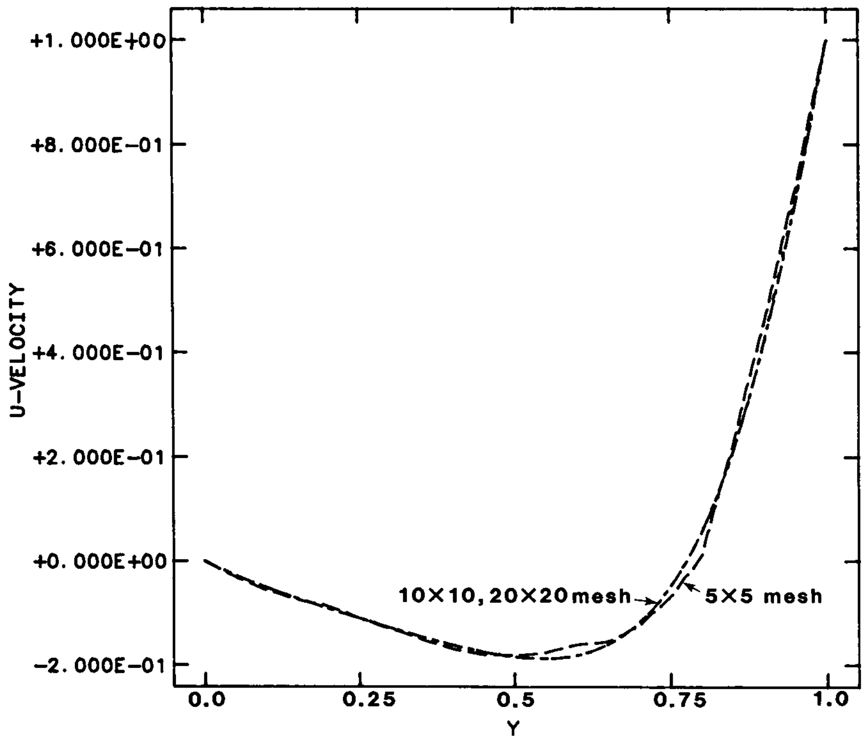


Figure 4. Horizontal velocity profiles at $x = 0{\cdot}8$ for 8-node element and 5×5, 10×10, 20×20 meshes

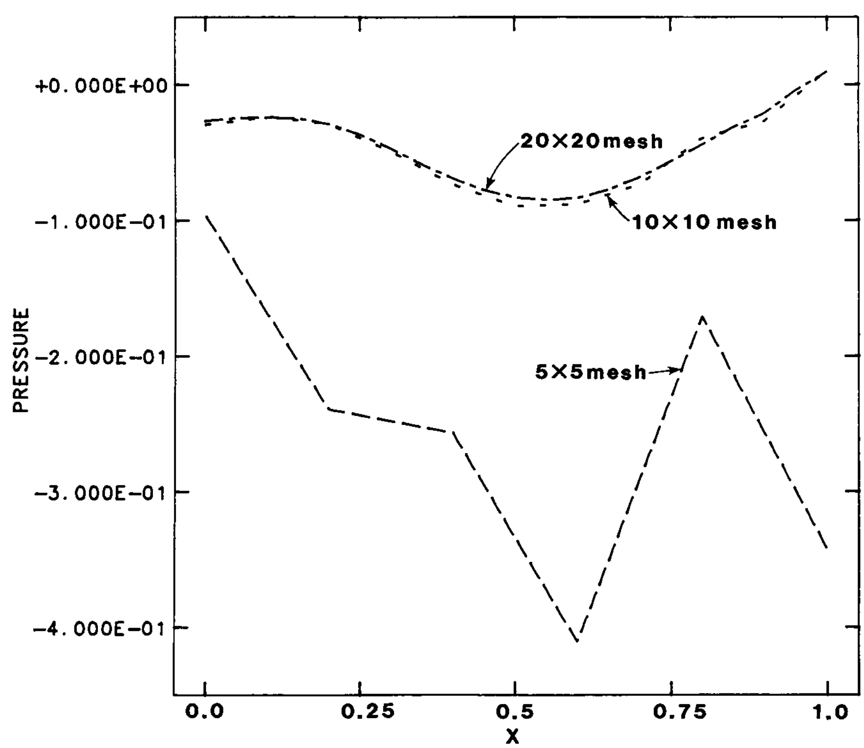
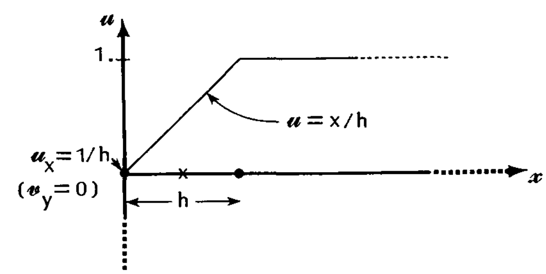


Figure 5. Pressure profiles along $y = 0{\cdot}8$ for flow at a higher Reynolds number, using the 8-node element and uniform meshes of (5×5), (10×10) and (20×20) elements

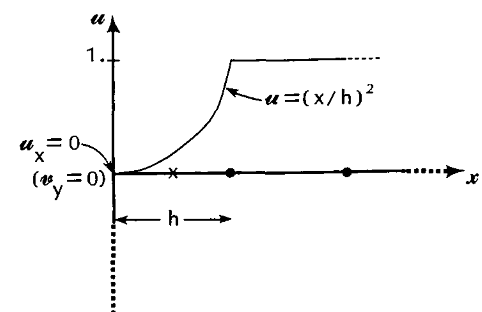
The coarse grid results are still inadequate, but the oscillations are not as pronounced as before. Results on the (10 × 10) and (20 × 20) grids are in good agreement and the boundary oscillation is no longer evident. Thus, the effect of the non-linear term is to mitigate the influence of the corner singularity. A similar behaviour was observed by Krishnan and Carey[3] for the penalty method using the bilinear element. Note, however, that the pressure error in the penalty solution has a spurious 'chequerboard' component. Researchers have categorized such modes previously for elements with discontinuous pressure bases.[4–6] No such mode is present for the mixed method using the 8-node element.[2]

It is instructive to examine the pressure values along a vertical side below the corner. For the (10 × 10) grid and $Re = 10$, with the first variant of the boundary condition (fixed slope), we

(a)



(b)



(c)

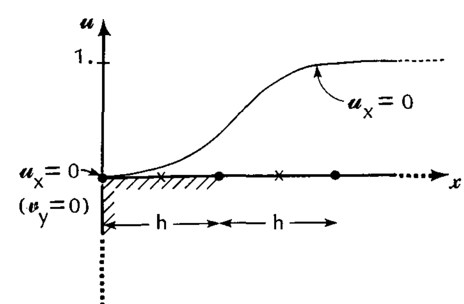


Figure 6. Smoothing boundary data (a) so that u' is zero at the corners in (b), and (c).

obtain the nodal values from top to bottom as $-7{\cdot}92$, $0{\cdot}311$, $-1{\cdot}3233$, $1{\cdot}29$, $-0{\cdot}265$, $0{\cdot}0322$, $-0{\cdot}008096$, $0{\cdot}05$, $0{\cdot}028$, $0{\cdot}0216$, $0{\cdot}0$. Note the sign change and pronounced oscillation near the top. We have found numerically that averaging the pressures at the element centroids provides a lower degree smoother projection. The pronounced local nature of the pressure oscillation suggests that the behaviour and explanation are different from that of elements previously analysed. We now offer an interpretation and modification which lead to improved results.

Physically, the pressures may be viewed as acting to enforce mass conservation ($\nabla \cdot \mathbf{u} = 0$). Hence, the loss of accuracy and presence of pronounced local oscillations reflect the inability of the approximation to satisfy mass conservation adequately. *Even though the velocity field in the approximation is continuous, the divergence of the velocity has a strong discontinuity at the upper corners.* Let us examine this observation in more detail. In the 'non-leaky' cavity models given earlier, the horizontal velocity of the lid is prescribed as varying linearly over a specified subinterval from zero at the upper corners to unity (Figure 6(a)). Now, on the vertical sides $v = 0$, so $v_y = 0$ at the upper corners. However, since the lid velocity varies linearly near the corners $u_x =$ constant here and, hence, $u_x + v_y \neq 0$ at the corners. This error in the divergence expression in the element is accommodated by a corresponding local pressure error. There is a similar, but lesser, jump in u_x at the interior node on the lid where $u = 1$ is attained. However, v_y is not specified here and, hence, the model can more reasonably approximate $u_x + v_y = 0$. Accordingly, the pressures are better behaved (e.g. see Figure 2).

These observations suggest that we select the fit to the boundary data in a slightly different manner, projecting the data to match $u_x = 0$ at the corners. To examine this, we consider two

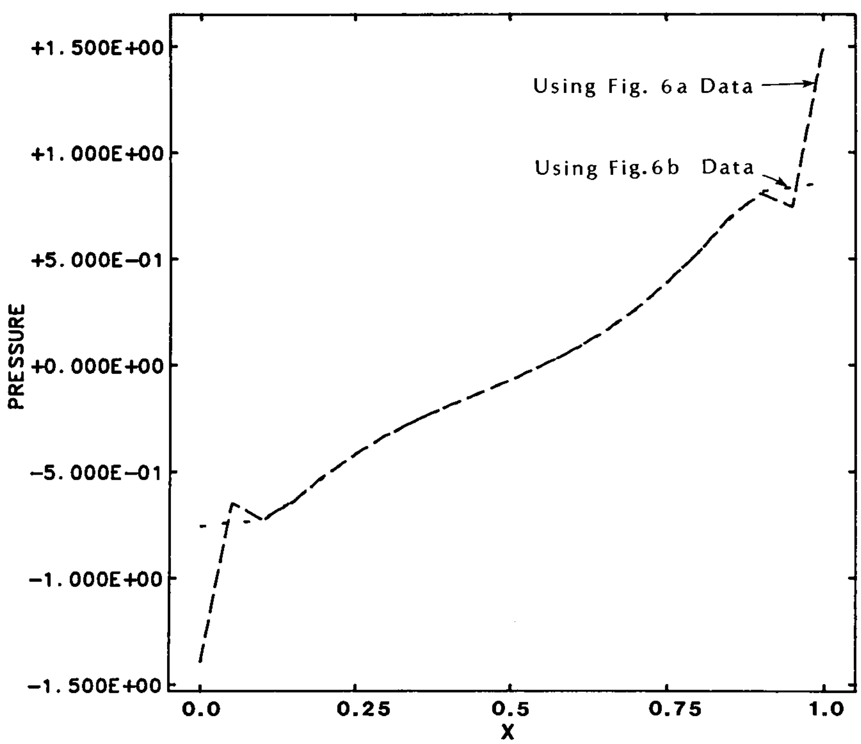


Figure 7. Pressure profiles for the 8-node element with data smoothing described in Figure 6(b) for 'varying slope' boundary condition (ii) and 20 × 20 mesh (compared to case for Figure 6(a))

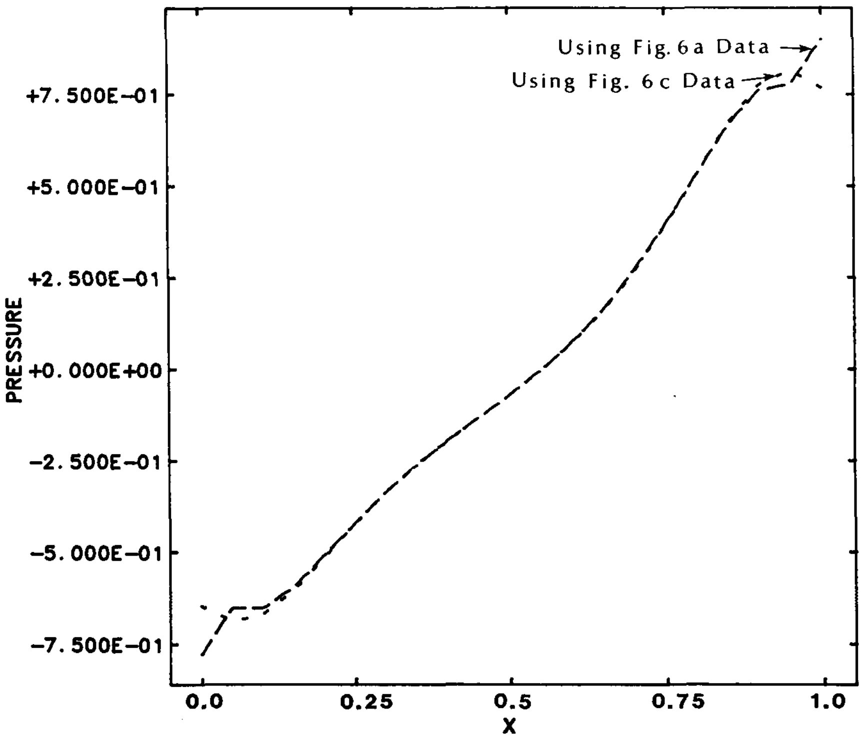


Figure 8. Pressure profiles for the 8-node element with data smoothing described in Figure 6(c) for 'Fixed slope' boundary condition (i) and 20 × 20 mesh (compared to case for Figure 6(a) with $u = 1$ on $0{\cdot}2 \leqslant x \leqslant 0{\cdot}8$)

modifications of the corner data. The first case (Figure 6(b)) corresponds to selecting a quadratic approximation to the lid velocity through (0,0) ($h/2$, 1/4), (h, 1) which implies that $u_x = 0$ at the corner $x = 0$ as shown. In the second case (Figure 6(c)), the quadratic approximation is selected over two elements (fitting 0, 1/8, 1/2, 7/8, 1 as shown) and $u_x = 0$ is satisfied at $x = 0$ with u_x continuous on $y = 1$. The pressures for these two cases are shown in Figures 7 and 8, respectively. Note that in the calculations for Figure 8 the data for $u(x)$ on the lid increase from zero to unity over the interval $0 \leqslant x \leqslant 0{\cdot}2$ (4 elements) whereas in Figure 7 the increase occurs across a single element ($0 \leqslant x \leqslant 0{\cdot}05$). This choice of data corresponds to that considered previously in the comparison of Figure 2. Clearly, the effect of regularizing the data is very pronounced, and the local pressure oscillations for the eight-node element are now almost absent. For comparison purposes, we also computed the solution using a quadratic approximation to u with $u_x = 0$ at $x = h$ where one fits (0,0) ($h/2$, 0·75) and (h, 1). The value of u_x at the corner is larger even than that in the previous linear cases, and the oscillations in computed pressure are worse, a result consistent with the present interpretation. For brevity, detailed numerical comparison results are not given here but are included in a report.[7]

CONCLUDING REMARKS

There is also a slight improvement in the quality of the pressures obtained with the 9-node element when the smoothed boundary data are used. However, the additional internal degree of freedom in the 9-node element suffices to allow adequate approximation of $u_x + v_y = 0$ near the corners even for the linear fit to data. Recalling that the 8-node 'serendipity' element is obtained by constraining the midside basis functions to vary linearly rather than quadratically in the direction normal to the side, derivatives will be less well approximated and the inadequacy in treating the corner error in $\nabla \cdot \mathbf{u}$ more pronounced.

Although it has little bearing on the present study of the 8-node element, for completeness and to satisfy our curiosity, we also computed the solution to the cavity problem using a penalty formulation with the 9-node element and 2×2 Gauss integration of the penalty term. This element is known to produce a superimposed chequerboard type pressure oscillation mode at the element Gauss points. We found, not surprisingly, that the pressure oscillation is still present when the 'smoothed' boundary data are used, but the amplitude of the oscillatory mode is much smaller. In effect, the error in $\nabla \cdot \mathbf{u}$ is smaller and, hence, there is less error projected into the spurious mode. This is also consistent with earlier observations that, if the data and solution are sufficiently well behaved, the spurious mode component may be negligible.[5]

In concluding, our approach of appropriately smoothing the data (by setting $u_x = 0$) is simple and effective in mitigating local pressure errors for the 8-node element and provides guidelines for projecting the boundary data, under which this element can be better used.

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REFERENCES

1. P. S. Huyakorn, C. Taylor, R. L. Lee and P. M. Gresho, 'A comparison of various mixed interpolated finite elements in velocity-pressure formulation of the Navier–Stokes equations', *Computers and Fluids*, **6**, 25–35 (1978).
2. R. L. Sani, P. M. Gresho, R. L. Lee and D.Griffiths, 'The cause and cure (?) of the spurious pressure generated by certain FEM solutions of the incompressible Navier–Stokes equations: Parts 1 and 2, *Int. j. numer. methods fluids*, **1**, 17–43, 171–204 (1981).
3. R. Krishnan and G. F. Carey, 'On the computed pressures for Navier–Stokes problems at increasing Reynolds numbers using the penalty finite element method', *Int. j. numer. methods fluids*, **5**, 439–442 (1985).
4. T. J. R. Hughes, W. K. Liu and A. Brooks, Finite element analysis of incompressible viscous flows by the penalty function formulation', *J. Comp. Phys.*, **30**, 1–60 (1979).
5. G. F. Carey and R. Krishnan, 'Penalty approximation of Stokes flow', *Comp. Meth. Appl. Mech. Eng.*, **35**, (2), 169–206 (1982).
6. J. T. Oden and O. P. Jacquotte, 'Stability of some mixed finite element methods for Stokesian flows', *Comp. Meth. Appl. Mech. and Eng.*, **43**, (2), 231–248 (1984).
7. G. F. Carey, R. Krishnan and R. McLay, 'Pressure oscillations for the driven cavity and smoothing of boundary conditions', *TICOM Report*, University of Texas at Austin, 1985.